Indian Statistical Institute
Mid-Semestral Examination
Differential Geometry I-MMath I

Time: 2 hours | Max. Marks : 40

Answer all questions. All claims must be accompanied by complete and correct justification(s).

(1) State whether the following statements are *True* or *False*. Justify.
  (a) Every vector field on $\mathbb{R}$ is complete.
  (b) There exists a 2-surface S in $\mathbb{R}^3$ whose spherical image consists of exactly 4 points.
  (c) If α is a constant speed parametrized curve in a surface S, then α is a geodesic.
  (d) $\mathbb{R}^2 - 0$ is geodesically complete. [$4 \times 4 = 16$]

(2) Show that a parametrized curve α in the unit n-sphere S^n is a geodesic if and only if it is of the form
$$\alpha(t) = (\cos at)e_1 + (\sin at)e_2$$
for some pair of orthonormal vectors $e_1, e_2 \in \mathbb{R}^{n+1}$ and $a \in \mathbb{R}$. [8]

(3) Discuss the definition of the Weingarten map. Let $\mathbb{X}, \mathbb{Y}$ be two smooth tangent vector fields on a n-surface S in $\mathbb{R}^{n+1}$ oriented by $\mathbb{N}$. Show that
$$\left(\nabla_{\mathbb{X}(p)}\mathbb{Y}\right) \cdot \mathbb{N}(p) = \left(\nabla_{\mathbb{Y}(p)}\mathbb{X}\right) \cdot \mathbb{N}(p)$$
for all $p \in S$. Further show that the vector field $[\mathbb{X}, \mathbb{Y}]$ defined by
$$[\mathbb{X}, \mathbb{Y}](p) = \left(\nabla_{\mathbb{X}(p)}\mathbb{Y}\right) - \left(\nabla_{\mathbb{X}(p)}\mathbb{Y}\right)$$
is a tangent vector field on S. [$2 + 6 + 3$]

(4) State and prove the Frenet formulas for a plane curve. [5]